Quantum Hamiltonian Complexity

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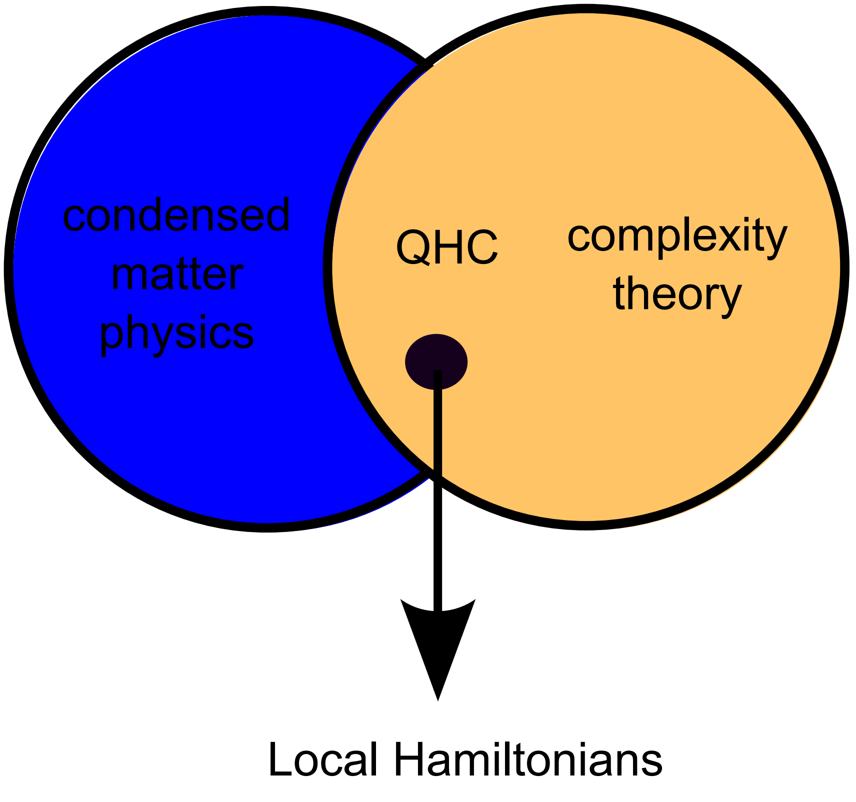
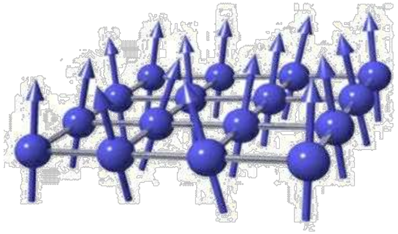


Local Hamiltonians



- Describe the interaction of quantum particles (spins) that sit on a lattice

$$H = \sum_X h_X$$

- $\langle\psi|H|\psi\rangle$ — the expectation of the energy of the state $|\psi\rangle$

- H determines the time evolution of the system via the Schrödinger equation:

$$|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle$$

- H determines the state of the system at **thermal equilibrium**

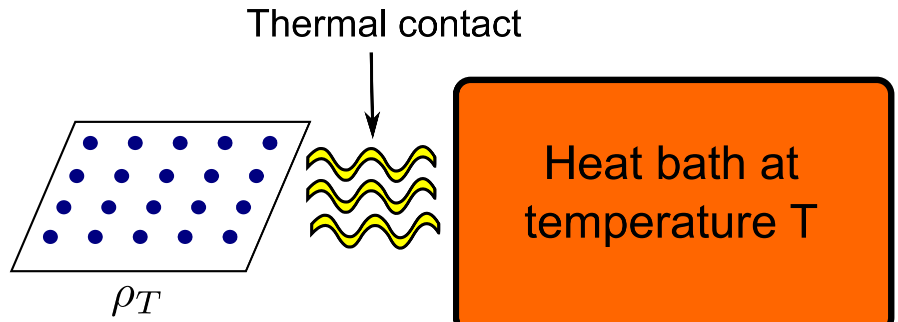
Thermal equilibrium

$$\rho_T \stackrel{\text{def}}{=} \frac{1}{Z} e^{-\frac{1}{T}H}$$

Gibbs state

$$Z \stackrel{\text{def}}{=} \mathrm{Tr}\, e^{-\frac{1}{T}H}$$

partition function



In the diagonalizing basis of H: $$\rho_T \stackrel{\text{def}}{=} \frac{1}{Z}\sum_i |\psi_i\rangle\langle\psi_i| e^{-\epsilon_i/T}$$

As $T \to 0$, we get $\rho_T \to |\psi_0\rangle\langle\psi_0|$

$|\psi_0\rangle$ – the state with the minimal energy – the **ground state**

⇒ The ground state is central in determining the physics of the system at $T \to 0$

⇒ The ground state is the global minimum of a set of local constraints

Much like a classical k-SAT system!

Main questions in quantum Hamiltonian complexity:

What is the complexity of:

- ☆ Approximating the ground energy
- ☆ Approximating the Gibbs state at temperature T (and local observables)
- ☆ Approximating the time evolution

Valuable insights into the physics of the systems:

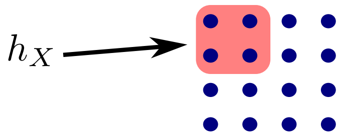
- structure of entanglement
- correlations
- phase transitions and criticality
- different phases of matter

Develop algorithms (classical and quantum) to study these systems

Formal definition:

- N particles sit on a D-dimensional lattice Λ
- Each particle lives in a d-dimensional Hilbert space ($d = 2$ unless specified otherwise)
- k-local Hamiltonian:

$$H = \sum_{X \subset \Lambda} h_X \quad |X| \leq k \text{ – nearest neighbors particles}$$



$$\|h_X\| \leq J$$

$$h_X = \hat{h}_X \otimes \mathbb{I}_{rest}$$

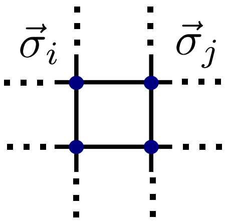
- Eigenvalues/Eigenvectors:

$$\epsilon_0 \leq \epsilon_1 \leq \epsilon_2 \leq \dots \qquad |\psi_0\rangle, |\psi_1\rangle, |\psi_2\rangle, \dots$$

- **Ground energy** and **Ground state:** ϵ_0 and $|\Omega\rangle = |\psi_0\rangle$
- Spectral gap: $\Delta\epsilon \stackrel{\text{def}}{=} \epsilon_1 - \epsilon_0$

Examples

Heisenberg model: $$H = -J \sum_{\langle i,j \rangle} \vec{\sigma}_i \cdot \vec{\sigma}_j + \vec{B} \cdot \sum_i \vec{\sigma}_i$$



$$\vec{\sigma}_i \cdot \vec{\sigma}_j \stackrel{\text{def}}{=} \sigma_i^x \cdot \sigma_j^x + \sigma_i^y \cdot \sigma_j^y + \sigma_i^z \cdot \sigma_j^z$$

Ising model w. transverse field: $$H = -J \sum_{\langle i,j \rangle} \sigma_i^z \cdot \sigma_j^z + B \sum_i \sigma_i^x$$

Local Hamiltonians as quantum generalizations of k-SAT forumas

Associate: energy $\leftrightarrow$ violations

Classical	Classical (quantum notation)	Quantum
Assignment: $s = (0,1,1,0,1,\ldots)$	$\|s\rangle = \|0,1,1,0,1,\ldots\rangle$	**Any** state $\|\psi\rangle$
local clause: $C_i = x_1 \vee \bar{x}_2 \vee x_3$ (rejects $(0,1,0)$)	Projector (in standard-basis) $Q_i \stackrel{\text{def}}{=} \|010\rangle\langle 010\|$	**Any** Hermitian term h_i on 3 qubits with bounded norm.
total # of violations of s	energy of $\|s\rangle$: $E_s = \langle s\|H\|s\rangle = \sum_i \langle s\|Q_i\|s\rangle$	energy of $\|\psi\rangle$: $E_\psi = \langle \psi\|H\|\psi\rangle = \sum_i \langle \psi\|h_i\|\psi\rangle$
minimizing assignment	ground state of $H = \sum_i Q_i$	ground state of $H = \sum_i h_i$
minimal # of violations	ground energy of $H = \sum_i Q_i$	ground energy of $H = \sum_i h_i$

The Local Hamiltonian Problem (LHP)

LHP

Given a local Hamiltonian $H = \sum_X h_X$, together with two numbers $b > a$ such that $b - a > \frac{1}{\text{poly}(N)}$, decide whether:

YES instance: $\epsilon_0 \leq a$

NO instance: $\epsilon_0 \geq b$

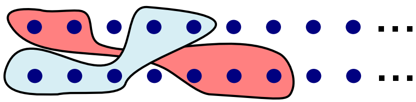
In other words: Find a $1/\text{poly}(N)$ approximation of ϵ_0

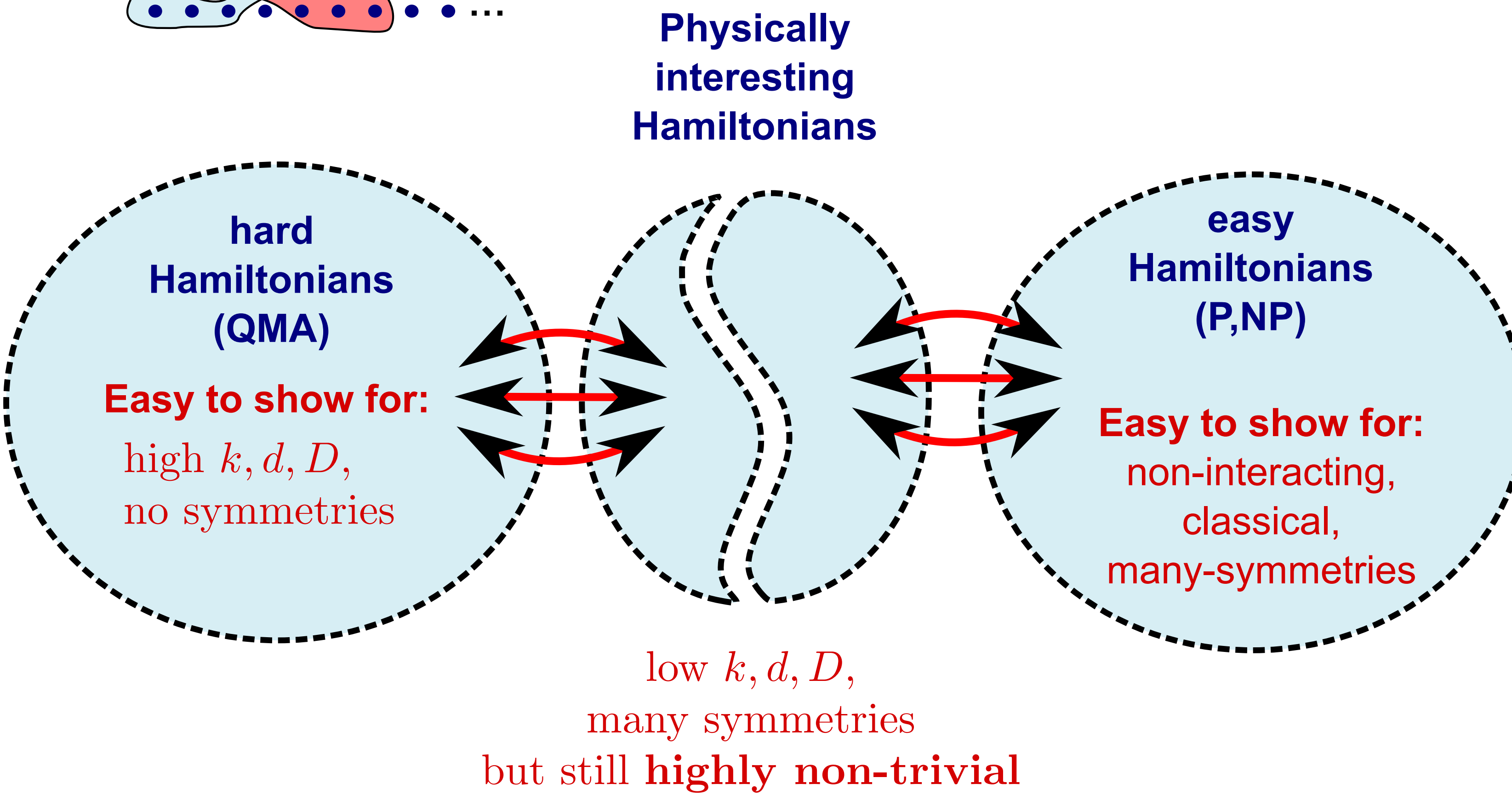
Central result: the "quantum Cook-Levin" theorem (Kitaev, '00)

The LHP with $k = 5$ is QMA complete (QMA = quantum NP)

Classifying the landscape of local Hamiltonians

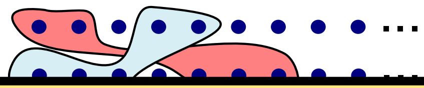
Kitaev's 5-local Hamiltonian:

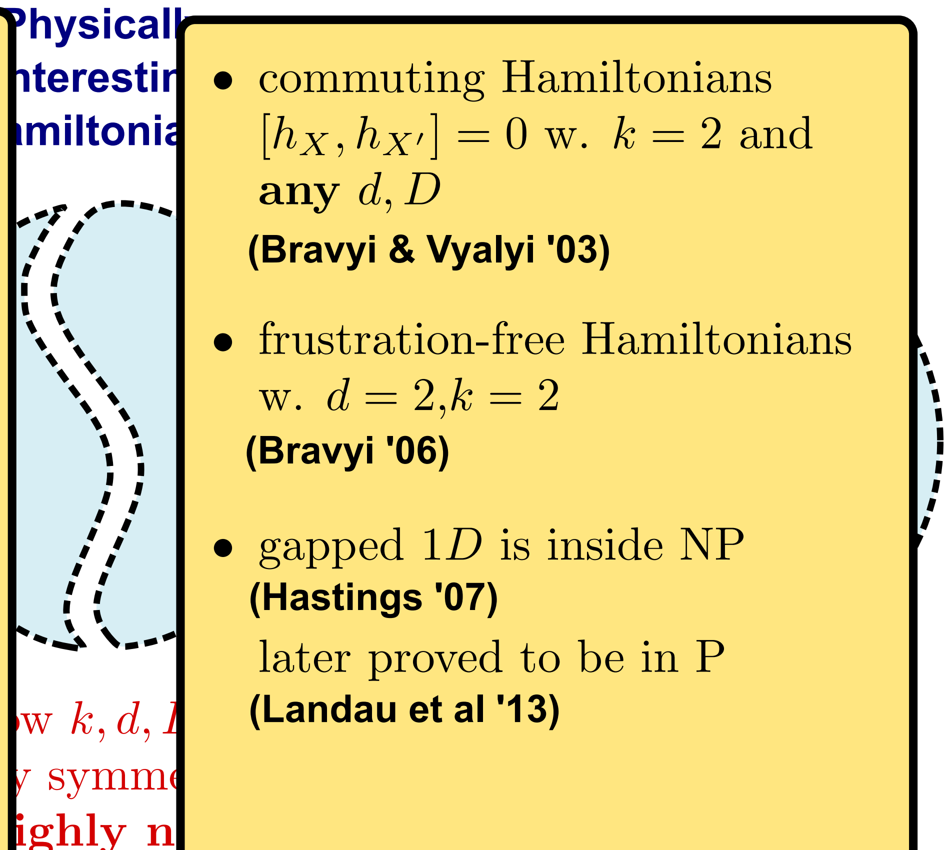




Classifying the landscape of local Hamiltonians

Kitaev's 5-local Hamiltonian:





- 2-local
  (Kempe, Kitaev & Regev '04)
- 2-local on a 2D lattice
  (Oliveira & Terhal '05)
- 2-local on a line w. $d = 12$)
  (Aharonov et. al. '07)
  later improved to $d = 8$
  (Hallgren et al '13)
- Heisenberg model on 2D lattice
  (Schuch \& Verstraete '07)
- Classification of all 2-local w. a fixed set of interactions
  (Cubitt & Montanaro, '13)

- commuting Hamiltonians $[h_X, h_{X'}] = 0$ w. $k = 2$ and **any** d, D
  (Bravyi & Vyalyi '03)
- frustration-free Hamiltonians w. $d = 2$,$k = 2$
  (Bravyi '06)
- gapped $1D$ is inside NP
  (Hastings '07)
  later proved to be in P
  (Landau et al '13)

A (bold) conjecture

The complexity of the **gapped** LHP (i.e., a spectral gap $\Delta\epsilon = \mathcal{O}(1)$) and constant d, k, D is classical:

- ⟹ The 1D case is in P
- ⟹ The 2D,3D, $\ldots$ cases are in NP

- ☆ In 1D this has been proved by Landau, Vazirani & Vidick '13
- ☆ In higher D the problem is wide open.

An intermediate outline

- ⇒ Why gaps matter: AGSPs
- ⇒ The detectability-lemma AGSP and the exponential decay of correlations
- ⇒ The Chebyshev AGSP and the 1D area-law
- ⇒ Matrix-Product-states, and why the 1D problem is inside NP
- ⇒ 1D algorithms
- ⇒ 2D and beyond: tensor-networks, PEPs and possible directions to proceed

The grand plan

To show that a class of LHP is inside NP (or P), we can try to show that the ground state $|\Omega\rangle$ admits an **efficient classical description:**

1. $|\Omega_c\rangle$ is described by poly(N) classical bits
2. $\langle\Omega_c|A|\Omega_c\rangle$ can be efficiently approximated up to $\|A\|/\text{poly}(N)$ for every local observable A
3. $|\langle\Omega_c|A|\Omega_c\rangle - \langle\Omega|A|\Omega\rangle| \leq \|A\|/\text{poly}(N)$

⇨ In such case we can simply use $|\Omega_c\rangle$ as a classical witness for the LHP problem since:

$$\langle\Omega|H|\Omega\rangle = \sum_X \langle\Omega|h_X|\Omega\rangle \simeq \sum_X \langle\Omega_c|h_X|\Omega_c\rangle = \langle\Omega_c|H|\Omega_c\rangle$$

local operators

Locality in ground states: AGSPs

How can we find an efficient classical description?

product state

$$|\phi_1\rangle \otimes |\phi_2\rangle \otimes \cdots \otimes |\phi_N\rangle$$

$\mathcal{O}(N)$ parameters

general state

$$\sum_{i_1 \ldots i_N} c_{i_1 \ldots i_N} |i_1 \ldots i_N\rangle$$

2^N parameters

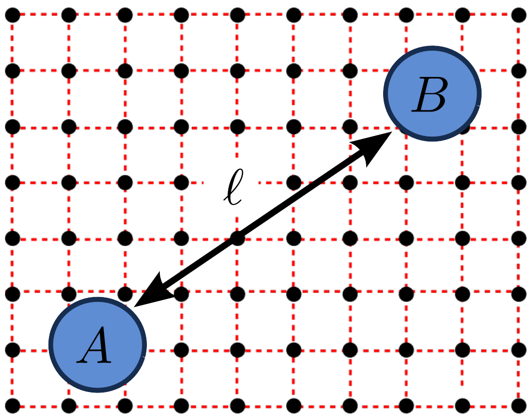
We need locality to bridge that gap

AGSP (**A**pproximate **G**round **S**pace **P**rojector)

An operator K is a δ-AGSP if:
$$\begin{cases} \bullet\ K|\Omega\rangle = |\Omega\rangle \\ \bullet\ \| K|\Omega^\perp\rangle \| \leq \delta \end{cases} \Rightarrow K = |\Omega\rangle\langle\Omega| + \mathcal{O}(\delta)$$

If K has a simple local structure then this could teach us about the local structure of $|\Omega\rangle$

Exponential decay of correlations



Exp' decay of correlations (Hastings '05)

In the G.S. of a gapped system the correlation function decays exponentially

$$\langle\Omega|AB|\Omega\rangle = \langle\Omega|A|\Omega\rangle\langle\Omega|B|\Omega\rangle + \|A\|\cdot\|B\|\cdot e^{-\ell/\ell_0}$$

$$\ell_0 \stackrel{\text{def}}{=} \mathcal{O}\left(\frac{V_{LB}}{\Delta\epsilon}\right)$$

We will use an **AGSP** to prove this for gapped **frustration-free** systems:

- $H = \sum_i Q_i$ (projectors)
- $Q_i|\Omega\rangle = 0$ (frustration freeness)
- Every Q_i touches at most $g = \mathcal{O}(1)$ other Q_j's (follows from constant D, k)

The detectability lemma

$$H = \sum_{i=1}^{M} Q_i \qquad K \stackrel{\text{def}}{=} (\mathbb{I} - Q_M)\cdot(\mathbb{I} - Q_{M-1})\cdots(\mathbb{I} - Q_1)$$

For any $|\psi\rangle$, let $|\phi\rangle \stackrel{\text{def}}{=} K|\psi\rangle$, and $\epsilon_\phi \stackrel{\text{def}}{=} \frac{1}{\|\phi\|^2}\langle\phi|H|\phi\rangle$.

$$\text{Then:} \quad \|\phi\|^2 \le \frac{1}{\epsilon_\phi/g^2 + 1}$$

Proof:

$$\langle\phi|H|\phi\rangle = \sum_i \langle\phi|Q_i|\phi\rangle$$

$$\langle\phi|Q_i|\phi\rangle = \langle\phi|Q_iQ_i|\phi\rangle = \|Q_i|\phi\rangle\|^2$$

$$\|Q_i|\phi\rangle\| = \|Q_i(\mathbb{I} - Q_M)\cdots(\mathbb{I} - Q_i)\cdots(\mathbb{I} - Q_1)|\psi\rangle\|$$

Assume $[Q_i, Q_j] \ne 0$:

$$\|(\mathbb{I} - Q_M)\cdots Q_i\cdot(\mathbb{I} - Q_j)\cdots(\mathbb{I} - Q_1)|\psi\rangle\| \le \|Q_i\cdot(\mathbb{I} - Q_j)\cdots(\mathbb{I} - Q_1)|\psi\rangle\|$$

$$\le \|Q_i\cdot Q_j\cdot(\mathbb{I} - Q_{j-1})\cdots(\mathbb{I} - Q_1)|\psi\rangle\| + \|Q_i\cdot(\mathbb{I} - Q_{j-1})\cdots(\mathbb{I} - Q_1)|\psi\rangle\|$$

$$\le \|Q_j\cdot(\mathbb{I} - Q_{j-1})\cdots(\mathbb{I} - Q_1)|\psi\rangle\| + \|Q_i\cdot(\mathbb{I} - Q_{j-1})\cdots(\mathbb{I} - Q_1)|\psi\rangle\|$$

$$\le \ldots \le \sum_{j:[Q_i,Q_j]\ne 0} \|Q_j\cdot(\mathbb{I} - Q_{j-1})\cdots(\mathbb{I} - Q_1)|\psi\rangle\|$$

$$\|Q_i|\phi\rangle\| \le \sum_{j:[Q_i,Q_j]\neq 0} \|Q_j \cdot (\mathbb{I}-Q_{j-1})\cdots(\mathbb{I}-Q_1)|\psi\rangle\|$$

$$\left(\sum_{i=1}^{g} x_i\right)^2 \le g\sum_{i=1}^{g} x_i^2$$

$$\Rightarrow \langle\phi|Q_i|\phi\rangle = \|Q_i|\phi\rangle\|^2 \le g \sum_{j:[Q_i,Q_j]\neq 0} \|Q_j \cdot (\mathbb{I}-Q_{j-1})\cdots(\mathbb{I}-Q_1)|\psi\rangle\|^2$$

$$\Rightarrow \langle\phi|H|\phi\rangle \le g^2 \sum_j \|Q_j \cdot (\mathbb{I}-Q_{j-1})\cdots(\mathbb{I}-Q_1)|\psi\rangle\|^2$$

telescopic sum

$$= g^2\left[1-\|(\mathbb{I}-Q_M)\cdots(\mathbb{I}-Q_1)|\psi\rangle\|^2\right] = g^2[1-\|\phi\|^2]$$

$$\Rightarrow \epsilon_\phi\|\phi\|^2 \le g^2(1-\|\phi\|^2)$$

$$\Rightarrow \|\phi\|^2 \le \frac{1}{\epsilon_\phi/g^2+1} \qquad \blacksquare$$

Conclusion:

When the system is frustration-free, K is a δ-AGSP with $\delta = \frac{1}{\sqrt{\Delta\epsilon/g^2+1}}$.

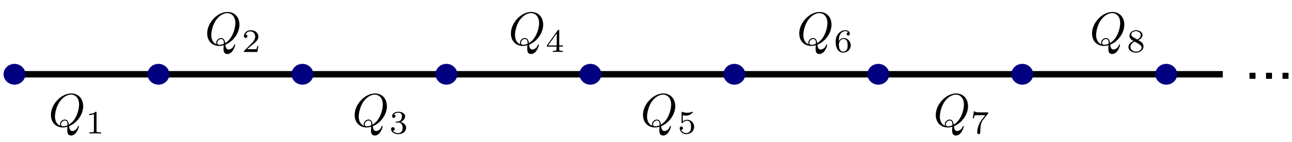
- $Q_i|\Omega\rangle = 0 \ \Rightarrow\ K|\Omega\rangle = (\mathbb{I}-Q_M)\cdots(\mathbb{I}-Q_1)|\Omega\rangle = |\Omega\rangle$
- For $|\Omega^\perp\rangle$, $\epsilon_{\Omega^\perp} \ge \epsilon_1 = \Delta\epsilon$. Therefore, by the D.L.:

$$\|K|\Omega^\perp\rangle\| \le \frac{1}{\sqrt{\Delta\epsilon/g^2+1}}$$

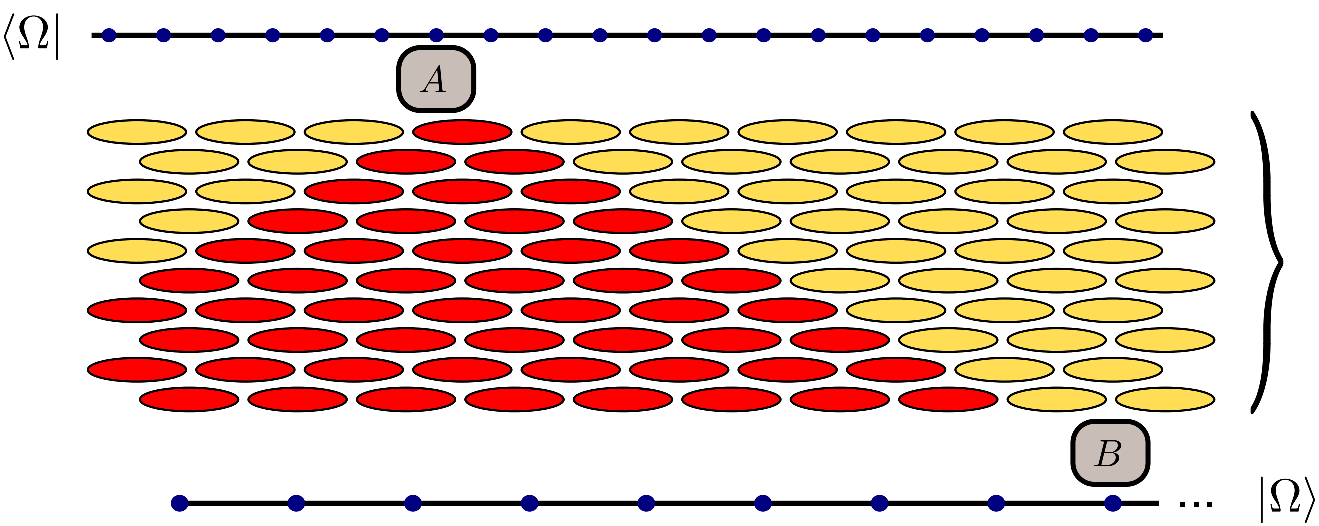
Exponential decay of correlations using the detectability-lemma AGSP

Even layer:

Odd layer:



$$K = \overbrace{(\mathbb{I} - Q_1)\cdot(\mathbb{I} - Q_3)\cdots}^{\text{odd layer}}\overbrace{(\mathbb{I} - Q_2)\cdot(\mathbb{I} - Q_4)\cdots}^{\text{even layer}}$$



\# of layers n proportional to distance ℓ

$\ell_0 = \mathcal{O}\left(\frac{1}{\Delta\epsilon g^2}\right)$

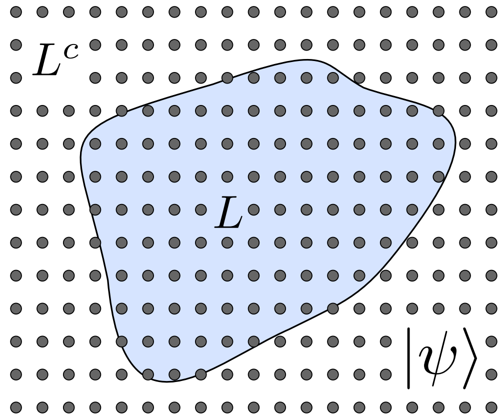
$$\langle\Omega|AB|\Omega\rangle = \langle\Omega|AK^nB|\Omega\rangle$$

but: $K^n = |\Omega\rangle\langle\Omega| + \delta^n \simeq |\Omega\rangle\langle\Omega| + e^{-\mathcal{O}\left(\ell\Delta\epsilon g^2\right)} \stackrel{\text{def}}{=} |\Omega\rangle\langle\Omega| + e^{-\ell/\ell_0}$

$$\Rightarrow \boxed{\langle\Omega|AB|\Omega\rangle \simeq \langle\Omega|A|\Omega\rangle\cdot\langle\Omega|B|\Omega\rangle + \|A\|\cdot\|B\|\cdot e^{-\ell/\ell_0}}$$

Area laws

Schmidt decomp': $|\psi\rangle = \sum_{i=1}^{R} \lambda_i |L_i\rangle \otimes |L_i^c\rangle$



Entanglement entropy:

$$S_L(\psi) \stackrel{\text{def}}{=} -\operatorname{Tr}\rho_L \ln \rho_L = -\sum_{i=1}^{R} \lambda_i^2 \ln(\lambda_i^2)$$

For **general** states, $S_L(\psi) \approx \mathcal{O}\left(|L|\right)$ **Volume law**

$|\psi\rangle$ must be described by $d^{|\mathcal{O}(|L|)}$ coefficients

For **special** states, $S_L(\psi) \approx \mathcal{O}\left(|\partial L|\right)$ **Area law**

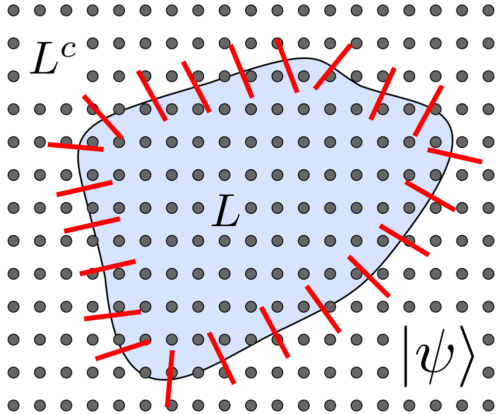
$|\psi\rangle$ can be described using only $d^{|\mathcal{O}(|\partial L|)}$ coefficients

Conjecture

Ground states of gapped local Hamiltonians on a lattice satisfy the area law

Intuitive explanation:

Exponential decay of correlations $\Rightarrow$ Only the degrees of freedom along the boundary ∂L are entangled

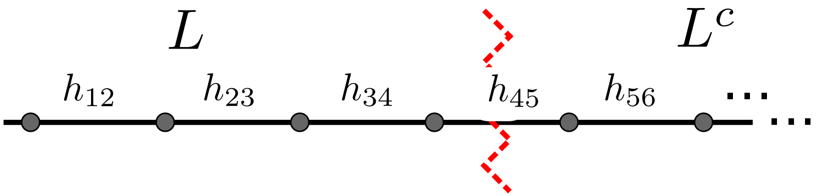


However,

So far, only the 1D case has been proved rigoursly (Hastings' 07)

An AGSP-based proof for the 1D area-law
(w. Aharonov, Kitaev, Landau & Vazirani)

The 1D area-law: $S_L(\Omega) \leq const$



Outline: $|L\rangle \otimes |R\rangle \to K|L\rangle \otimes |R\rangle \to K^2|L\rangle \otimes |R\rangle \to \ldots \to |\Omega\rangle$

Our main object:

(D, δ)-AGSP

- $K|\Omega\rangle = |\Omega\rangle$
- $\| K|\Omega^{\perp}\rangle \| \leq \delta$
- $K = \sum_{i=1}^{D} K_i^L \otimes K_i^R$

Assume:

We have (D,δ)-AGSP, and $|L\rangle \otimes |R\rangle$ such that $\mu = |\langle L \otimes R|\Omega\rangle| = \mathcal{O}(1)$

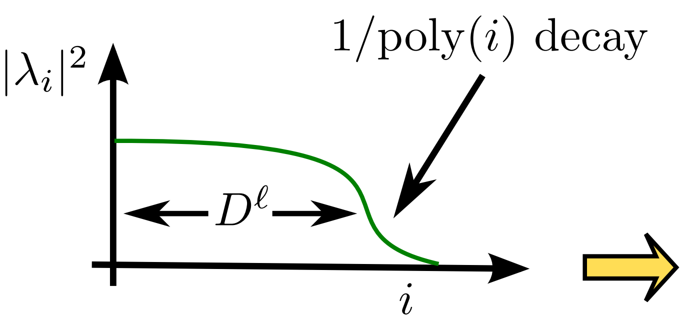
$$|\Omega\rangle = \mu|L\rangle \otimes |R\rangle + (1-\mu^2)^{1/2}|\Omega^{\perp}\rangle$$

(D,δ)-AGSP

- $K|\Omega\rangle = |\Omega\rangle$
- $\|K|\Omega^{\perp}\rangle\| \leq \delta$
- $K = \sum_{i=1}^{D} K_i^L \otimes K_i^R$

Then: Applying K^{ℓ} with $\ell = \mathcal{O}(\log\mu/\log\delta)$ will give a good approx to $|\Omega\rangle$

$$|\Omega\rangle = \sum_i \lambda_i |L_i\rangle \otimes |R_i\rangle$$



$$S_L(\Omega) \approx \mathcal{O}(\ell \cdot \log D) = \mathcal{O}(\log D \cdot \log\mu/\log\delta)$$

The bootstrapping lemma

Lemma

If there exists a (D,δ)-AGSP with $D\delta^2 < 1/2$ then there exists $|L\rangle \otimes |R\rangle$ with $\mu = |\langle L \otimes R|\Omega\rangle| \geq \frac{1}{\sqrt{2D}}$

Proof:

Let $|L\rangle \otimes |R\rangle$ be the product state with the largest overlap.

$|\phi\rangle \stackrel{\text{def}}{=} K|L\rangle \otimes |R\rangle = \sum_{i=1}^{D} \lambda_i |L_i\rangle \otimes |R_i\rangle \quad$ (Schmidt decomp' of $|\phi\rangle$)

Then on the one hand:

$$|\langle\Omega|\phi\rangle| \leq \sum_{i=1}^{D} \lambda_i |\langle\Omega|L_i \otimes R_i\rangle| \leq \mu \sum_{i=1}^{D} \lambda_i \leq \mu\sqrt{D}\sqrt{\textstyle\sum_i \lambda_i^2} = \mu\sqrt{D}\cdot\|\phi\| \quad (1)$$

On the other hand:

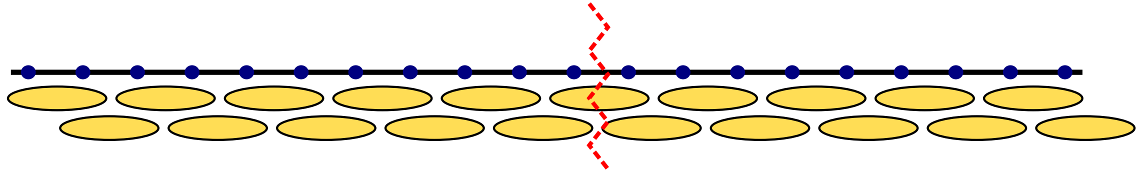
$$|L\rangle \otimes |R\rangle = \mu|\Omega\rangle \otimes |R\rangle + (1-\mu^2)^{1/2}|\Omega^\perp\rangle \quad \Rightarrow \quad |\phi\rangle = \mu|\Omega\rangle + (1-\mu^2)^{1/2} K|\Omega^\perp\rangle$$

$$\Rightarrow \ |\langle\Omega|\phi\rangle| = \mu \text{ and } \|\phi\| \leq \sqrt{\mu^2 + \delta^2}$$

Plugging into (1), we get: $\mu^2 \geq \frac{1}{D}(1 - D\delta^2) \geq \frac{1}{2D}$ ■

Good AGSPs are hard to find...

The detectability lemma
AGSP



Only one projector increases the S.R., but still... $D = d^2 \quad \delta = \frac{1}{\sqrt{\Delta\epsilon/g^2+1}} \quad (g=2)$

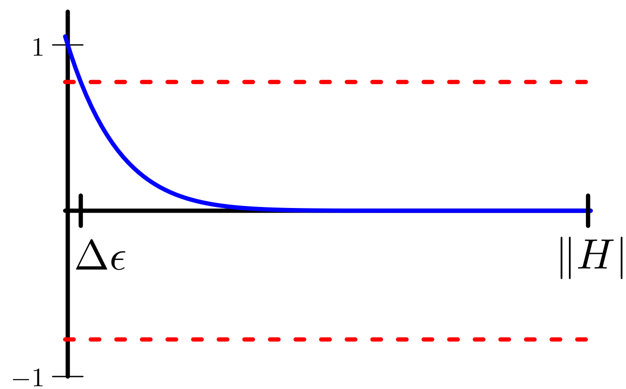
A different approach: Use a low-degree polynomial of H: $K \stackrel{\text{def}}{=} \text{poly}_q(H)$

Example:

$$\text{poly}_q(x) = \left(1 - \frac{x-\epsilon_0}{\|H\|-\epsilon_0}\right)^\ell$$

$$K \stackrel{\text{def}}{=} \left(\mathbb{I} - \frac{H-\epsilon_0}{\|H\|-\epsilon_0}\right)^q$$

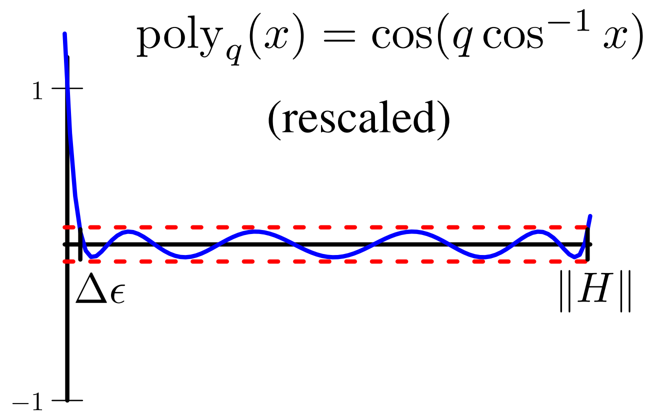
$$\delta = \left(1 - \frac{\Delta\epsilon}{\|H\|}\right)^q \simeq e^{-q\frac{\Delta\epsilon}{\|H\|}}$$



Can we do better?

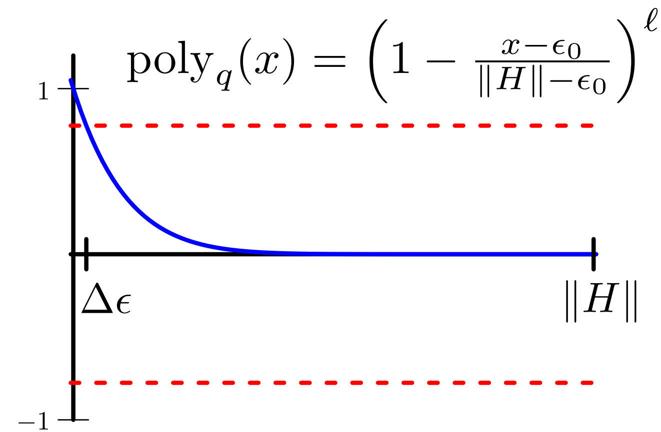
Chebyshev Polynomial

$$\delta \simeq e^{-2q\sqrt{\frac{\Delta\epsilon}{\|H\|}}}$$



Compare with:

$$\delta = \left(1 - \frac{\Delta\epsilon}{\|H\|}\right)^q \simeq e^{-q\frac{\Delta\epsilon}{\|H\|}}$$



Other ingredients in the proof

- We can truncate the upper spectrum of H to t, introducing only an error of $e^{-O(t)}$ to the ground state and ground energy

$$\delta \simeq e^{-2q\sqrt{\frac{\Delta\epsilon}{\|H\|}}} \to e^{-2q\sqrt{\frac{\Delta\epsilon}{t}}}$$

- Schmidt rank:

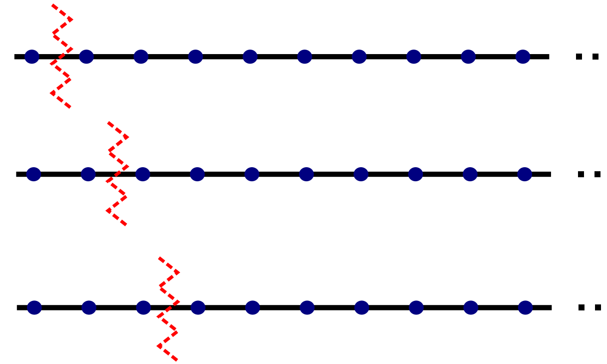
  One can write $H^q = \sum_{i=1}^{R} H_i^{(L)} \otimes H_i^{(R)}$ with $R = d^{\mathcal{O}\left(\sqrt{q}\right)}$

Taking all these points together, one constructs a Chebyshev-based AGSP with $D\delta^2 < \frac{1}{2}$

Constructing a Matrix-Product-State (MPS)

$$
\begin{aligned}
|\Omega\rangle &= \textstyle\sum_\alpha \lambda_\alpha^{[1]} |L_\alpha^{[1]}\rangle \otimes |R_\alpha^{[1]}\rangle \\
&= \textstyle\sum_\alpha \lambda_\alpha^{[2]} |L_\alpha^{[2]}\rangle \otimes |R_\alpha^{[2]}\rangle \\
&= \textstyle\sum_\alpha \lambda_\alpha^{[3]} |L_\alpha^{[3]}\rangle \otimes |R_\alpha^{[3]}\rangle \\
&\vdots
\end{aligned}
$$

we can truncate at each cut



Canonical MPS: (Vidal '03)

$$|\Omega\rangle = \sum_{i_1 \dots i_N} c_{i_1 \dots i_N} |i_1 \dots i_N\rangle$$

Iteratively express $|R_\alpha^{[j]}\rangle$ in terms of $|j\rangle \otimes |R_\beta^{[j+1]}\rangle$

$$\Rightarrow \quad c_{i_1 \dots i_N} = \sum_{\alpha_1, \alpha_2, \dots} \Gamma_{\alpha_1}^{[1] i_1} \cdot \lambda_{\alpha_1}^{[1]} \cdot \Gamma_{\alpha_1 \alpha_2}^{[2] i_2} \cdot \lambda_{\alpha_2}^{[2]} \cdot \Gamma_{\alpha_2 \alpha_3}^{[3] i_2} \cdot \lambda_{\alpha_3}^{[3]} \cdot \Gamma_{\alpha_3 \alpha_4}^{[4] i_2} \cdots$$

Taking only the first poly(N) largest α indices: $\quad |\Omega\rangle \to |\Omega_c\rangle = |\Omega\rangle + \frac{1}{\text{poly}(N)}$

This is a poly(N) description. But is it also efficient?

MPS as tensor-networks

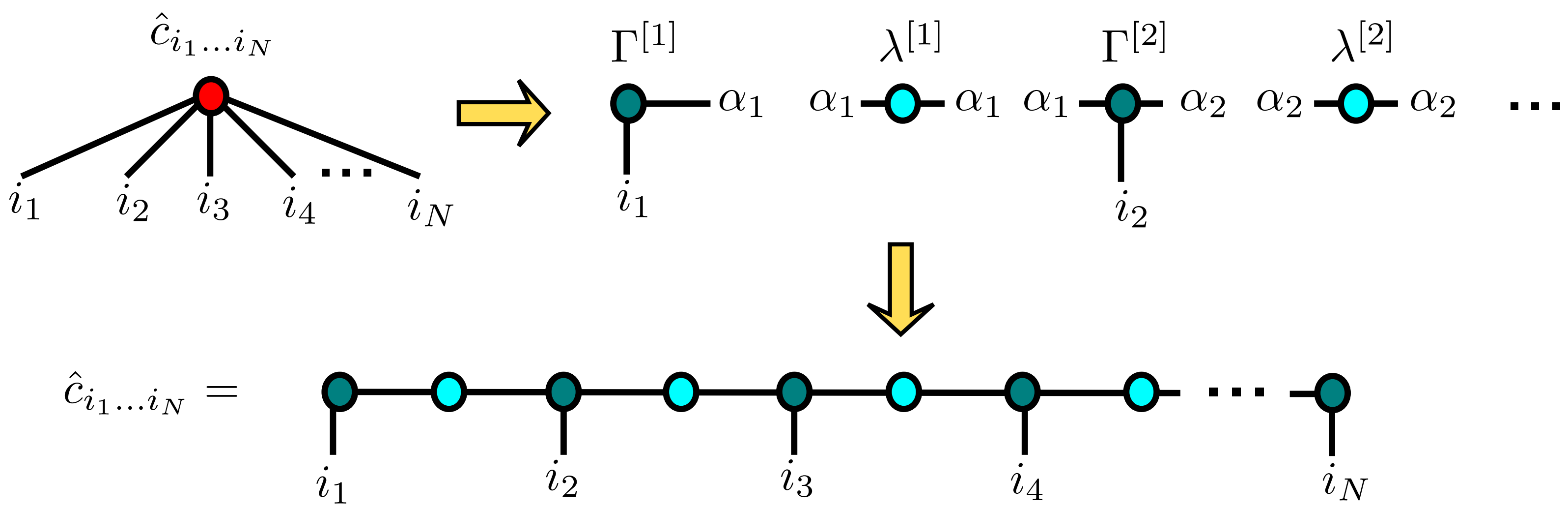
Can we efficiently calculate $\langle \Omega_c | A | \Omega_c \rangle$ for local observables?

$$|\Omega_c\rangle = \sum_{i_1 \ldots i_N} \hat{c}_{i_1 \ldots i_N} |i_1 \ldots i_N\rangle \qquad \hat{c}_{i_1 \ldots i_N} = \sum_{\substack{\alpha_1, \alpha_2, \ldots \\ \leq \mathrm{poly}(N)}} \Gamma^{[1] i_1}_{\alpha_1} \cdot \lambda^{[1]}_{\alpha_1} \cdot \Gamma^{[2] i_2}_{\alpha_1 \alpha_2} \cdot \lambda^{[2]}_{\alpha_2} \cdots$$

Tensor-network: vertices ↔ tensors

edges ↔ indices

connected edges ↔ contracted indices.



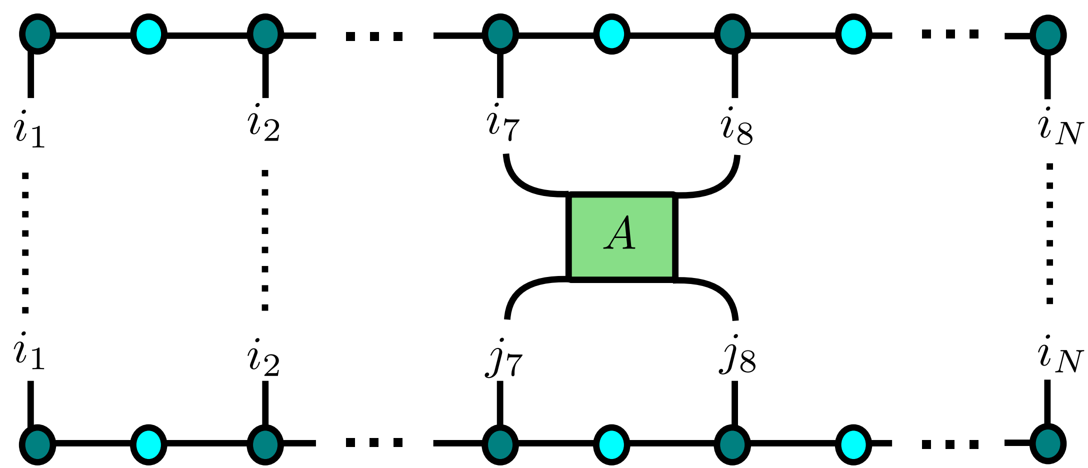
Calculating with MPS

Suppose we want to calculate $\langle \Omega_c | A | \Omega_c \rangle$, where A defined on particles 7,8

$$|\Omega_c\rangle = \sum_{i_1 \dots i_N} \hat{c}_{i_1 \dots i_N} |i_1 \dots i_N\rangle \qquad \langle \Omega_c| = \sum_{i_1 \dots i_N} \langle i_1 \dots i_N | \hat{c}^\dagger_{i_1 \dots i_N}$$

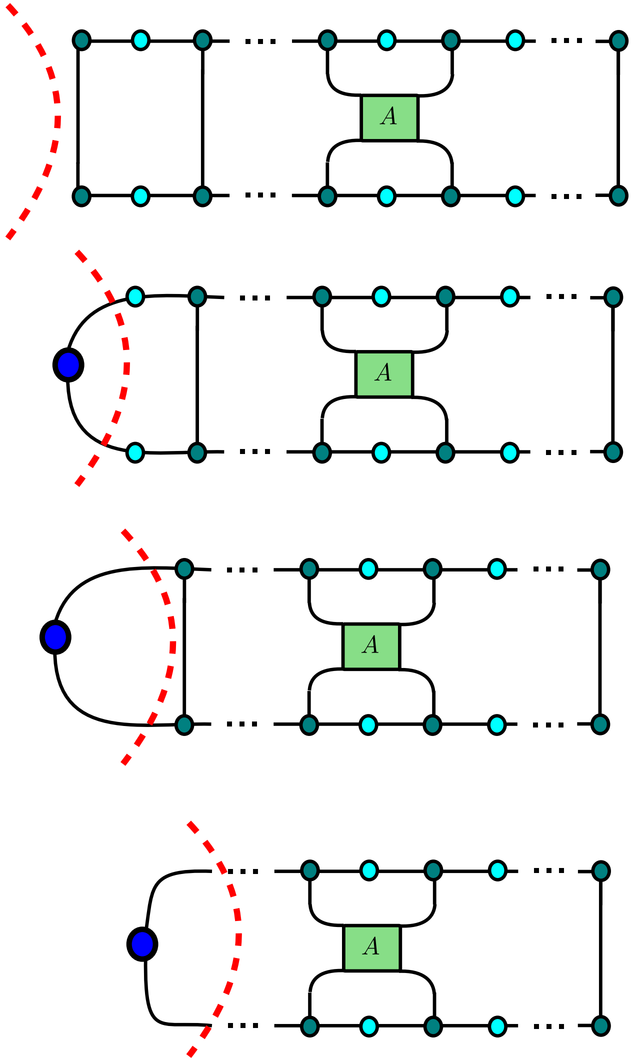
$$A = \sum_{\substack{i_7, i_8 \\ i_7, j_8}} \langle i_7, i_8 | A | j_7, j_8 \rangle \cdot |i_7, i_8\rangle\langle j_7, j_8| \overset{\text{def}}{=} \sum_{\substack{i_7, i_8 \\ i_7, j_8}} A^{i_7, i_8}_{j_7, j_8} \cdot |i_7, i_8\rangle\langle j_7, j_8|$$

$\langle \Omega_c | A | \Omega_c \rangle =$



Contracting a tensor-network: the swallowing bubble

$\langle \Omega_c | A | \Omega_c \rangle =$



At every step of the algorithm the bubble only cuts a constant number of edges, whose total indices span over at most a **polynomial r**ange

Calculating a local observable of an MPS can be done efficiently!

Summary of the 1D is inside NP argument

- When the system is gapped, at any cut along the chain the Schmidt coefficients decay polynomially after $i \geq \mathcal{O}\left(D^{\ell}\right) = const$

  ⇒ The system satisfies an area-law: $S(\Omega) \leq const$

  ⇒ We can truncate the Schmidt coefficients after $\text{poly}(N)$ to get a $1/\text{poly}(N)$ approximation for $|\Omega\rangle$

- From the truncation of the Schmidt coefficients we get a polynomial MPS

- Expectation values of the MPS can be efficiently calculated

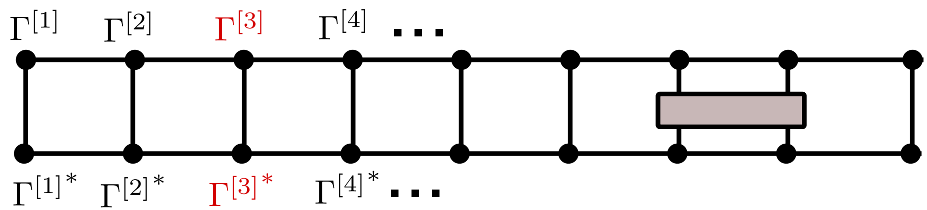
⇓

The MPS can be used as a classical witness to show that 1D gapped LHP is inside NP

Algorithms for finding the g.s. of gapped 1D systems

Density Matrix Renormalization Group (DMRG) (White '92)

Equivalent for locally optimizing the MPS (Rommer & Ostlund '96)



$$\langle\psi|H|\psi\rangle = \sum_X \langle\psi|h_X|\psi\rangle \textbf{ quadratic} \text{ in } \Gamma^{[i]}$$

TEBD (Vidal '03)

Approach the ground state by applying $e^{-\tau H}$ to an MPS

$$e^{-\tau H}|\psi\rangle \to |\Omega\rangle$$

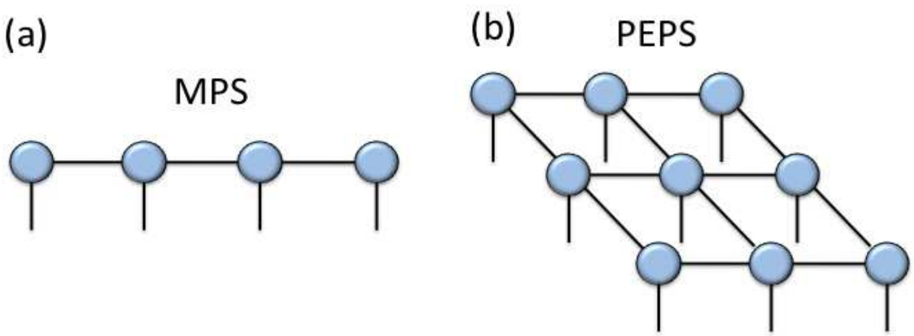
At every step the SR of the MPS increases, hence we truncate it to keep the MPS small

Dynamical programing (Landau, Vazirani & Vidick '13)

A random algorithm that **rigrously** converges to the g.s. with high probability. Based on applying Dynamical Programming to MPSs

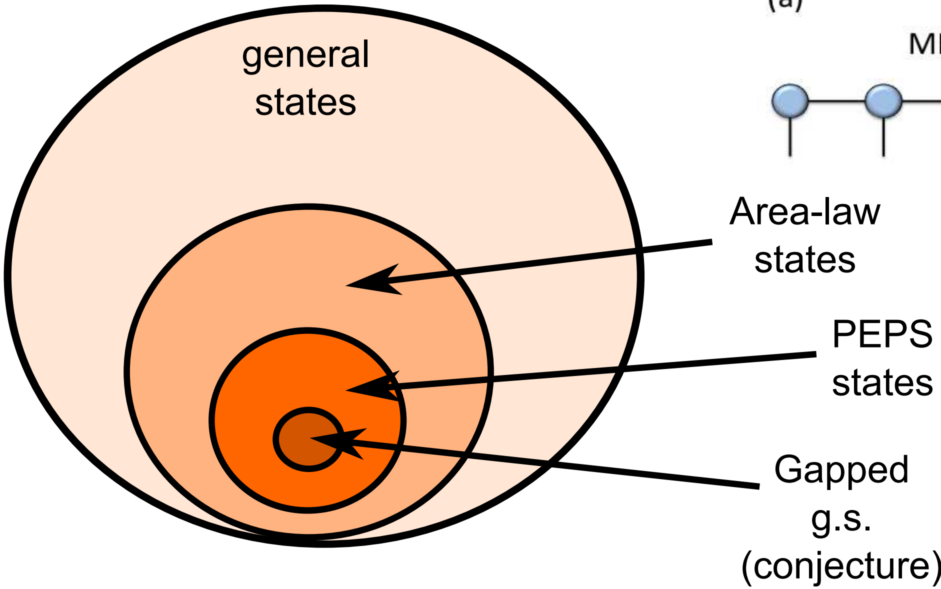
2D and beyond

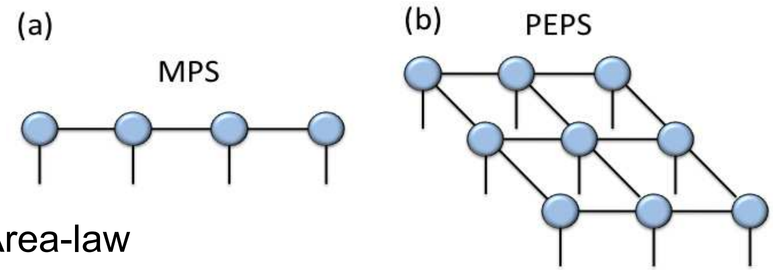
- We cannot hope for an efficient problem because already the classical problem (SAT in 2D) is NP hard
- However, by finding an efficient classical representation we may revolutionize the field like DMRG did in 1D
- Current approaches: use 2D tensor networks such as PEPS



(taken from Orùs '13)

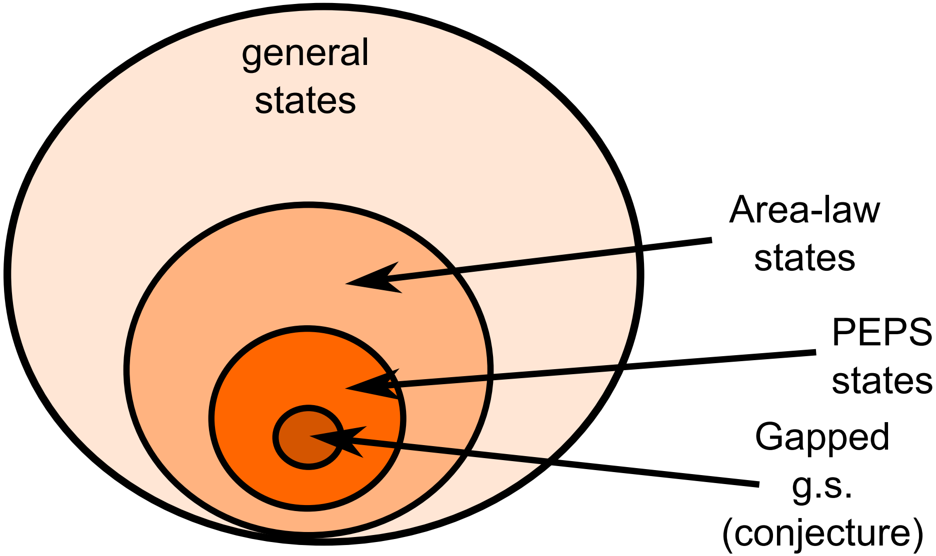
The difficulties in 2D





- PEPS states naturally satisfy the 2D area-law. However, the 2D area-law proof is still missing...
- Even if we had a 2D area-law proof, it would still not prove that the g.s. is well-approximated by a PEPS
- Even if the g.s. was known to be approximated by a PEPS, it is still not clear how to efficiently compute local observables with PEPS

But there's hope



- A 2D area-law proof would (if found) surely teach us much more about the structure of the g.s. than merely the area-law itself.

- Contracting a PEPS exactly is #P hard. However, we are not fully using the fact that we are interested in very special PEPS: **Those that represent gapped g.s. Some numerical evidences suggest that this can be done efficiently (Cirac et al '11)**

  There is much more structure (i.e., exp' decay of correlations), which can be used to prove efficient contraction.

Thank you !